

Asymmetric Quantum LDPC Codes

Pradeep Kiran Sarvepalli
Department of Computer Science
Texas A&M University
College Station, TX 77843
Email: pradeep@cs.tamu.edu

Martin Rötteler
NEC Laboratories America
4, Independence Way, Suite 200
Princeton, NJ 08540
Email: mroetteler@nec-labs.com

Andreas Klappenecker
Department of Computer Science
Texas A&M University
College Station, TX 77843
Email: klappi@cs.tamu.edu

Abstract—Recently, quantum error-correcting codes were proposed that capitalize on the fact that many physical error models lead to a significant asymmetry between the probabilities for bit flip and phase flip errors. An example for a channel which exhibits such asymmetry is the combined amplitude damping and dephasing channel, where the probabilities of bit flips and phase flips can be related to relaxation and dephasing time, respectively. We give systematic constructions of asymmetric quantum stabilizer codes that exploit this asymmetry. Our approach is based on a CSS construction that combines BCH and finite geometry LDPC codes.

I. INTRODUCTION

In many quantum mechanical systems the mechanisms for the occurrence of bit flip and phase flip errors are quite different. In a recent paper Ioffe and Mézard [10] postulated that quantum error-correction should take into account this asymmetry. The main argument given in [10] is that most of the known quantum computing devices have relaxation times (T_1) that are around 1 – 2 orders of magnitude larger than the corresponding dephasing times (T_2). In general, relaxation leads to both bit flip and phase flip errors, whereas dephasing only leads to phase flip errors. This large asymmetry between T_1 and T_2 suggests that bit flip errors occur less frequently than phase flip errors and a well designed quantum code would exploit this asymmetry of errors to provide better performance. In fact, this observation and its consequences for quantum error correction, especially quantum fault tolerance, have prompted investigations from various other researchers [1], [8], [20].

Our goal will be as in [10] to construct asymmetric quantum codes for quantum memories and at present we do not consider the issue of fault tolerance. We first quantitatively justify how noise processes, characterized in terms of T_1 and T_2 , lead to an asymmetry in the bit flip and phase flip errors. As a concrete illustration of this we consider the amplitude damping and dephasing channel. For this channel we can compute the probabilities of bit flip and phase flips in closed form. In particular, by giving explicit expressions for the ratio of these probabilities in terms of the ratio T_1/T_2 , we show how the channel asymmetry arises.

After providing the necessary background, we give two systematic constructions of asymmetric quantum codes based on BCH and LDPC codes, as an alternative to the randomized construction of [10].

II. BACKGROUND

Recall that a quantum channel that maps a state ρ to

$$(1 - p_x - p_y - p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z, \quad (1)$$

with $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is called a *Pauli channel*. For a Pauli channel, one can respectively determine the probabilities p_x, p_y, p_z that an input qubit in state ρ is subjected to a Pauli X, Y , or Z error.

A combined *amplitude damping and dephasing channel* \mathcal{E} with relaxation time T_1 and dephasing time T_2 that acts on a qubit with density matrix $\rho = (\rho_{ij})_{i,j \in \{0,1\}}$ for a time t yields the density matrix

$$\mathcal{E}(\rho) = \begin{bmatrix} 1 - \rho_{11}e^{-t/T_1} & \rho_{01}e^{-t/T_2} \\ \rho_{10}e^{-t/T_2} & \rho_{11}e^{-t/T_1} \end{bmatrix}.$$

This channel is interesting as it models common decoherence processes fairly well. We would like to determine the probability p_x, p_y , and p_z such that an X, Y , or Z error occurs in a combined amplitude damping and dephasing channel. However, it turns out that this question is not well-posed, since \mathcal{E} is not a Pauli channel, that is, it cannot be written in the form (1). However, we can obtain a Pauli channel \mathcal{E}_T by a technique called *twirling* [7], [5]. In our case, the twirling consists of conjugating the channel \mathcal{E} by Pauli matrices and averaging over the results. The resulting channel \mathcal{E}_T is called the *Pauli-twirl* of \mathcal{E} and is explicitly given by

$$\mathcal{E}_T(\rho) = \frac{1}{4} \sum_{A \in \{\mathbf{1}, X, Y, Z\}} A^\dagger \mathcal{E}(A\rho A^\dagger) A.$$

Theorem 1: Given a combined amplitude damping and dephasing channel \mathcal{E} as above, the associated Pauli-twirled channel is of the form

$$\mathcal{E}_T(\rho) = (1 - p_x - p_y - p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z,$$

where $p_x = p_y = (1 - e^{-t/T_1})/4$ and $p_z = 1/2 - p_x - \frac{1}{2}e^{-t/T_2}$. In particular,

$$\frac{p_z}{p_x} = 1 + 2 \frac{1 - e^{t/T_1(1-T_1/T_2)}}{e^{t/T_1} - 1}.$$

If $t \ll T_1$, then we can approximate this ratio as $2T_1/T_2 - 1$.

Proof: The Kraus operator decomposition [18] of \mathcal{E} is

$$\mathcal{E}(\rho) = \sum_{k=0}^2 A_k \rho A_k^\dagger, \quad (2)$$

where $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda-\gamma} \end{bmatrix}$; $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}$; $A_2 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$, and $\sqrt{1-\gamma-\lambda} = e^{-t/T_2}$, $1-\gamma = e^{-t/T_1}$. We can rewrite the Kraus operators A_i as

$$A_0 = \frac{1 + \sqrt{1-\lambda-\gamma}}{2} \mathbf{1} + \frac{1 - \sqrt{1-\lambda-\gamma}}{2} Z,$$

$$A_1 = \frac{\sqrt{\lambda}}{2} \mathbf{1} - \frac{\sqrt{\lambda}}{2} Z, \quad A_2 = \frac{\sqrt{\gamma}}{2} X - \frac{\sqrt{\gamma}}{2i} Y.$$

Rewriting $\mathcal{E}(\rho)$ in terms of Pauli matrices yields

$$\begin{aligned} \mathcal{E}(\rho) &= \frac{2-\gamma+2\sqrt{1-\lambda-\gamma}}{4} \rho + \frac{\gamma}{4} X \rho X + \frac{\gamma}{4} Y \rho Y \\ &+ \frac{2-\gamma-2\sqrt{1-\lambda-\gamma}}{4} Z \rho Z \\ &- \frac{\gamma}{4} \mathbf{1} \rho Z - \frac{\gamma}{4} Z \rho \mathbf{1} + \frac{\gamma}{4i} X \rho Y - \frac{\gamma}{4i} Y \rho X. \end{aligned} \quad (3)$$

It follows that the Pauli-twirl channel \mathcal{E}_T is of the claimed form, see [5, Lemma 2]. Computing the ratio p_z/p_x we get

$$\begin{aligned} \frac{p_z}{p_x} &= \frac{2-\gamma-2\sqrt{1-\lambda-\gamma}}{\gamma} = \frac{1+e^{-t/T_1}-2e^{-t/T_2}}{1-e^{-t/T_1}}, \\ &= 1 + 2 \frac{e^{-t/T_1}-e^{-t/T_2}}{1-e^{-t/T_1}} = 1 + 2 \frac{1-e^{t/T_1-t/T_2}}{e^{t/T_1}-1} \\ &= 1 + 2 \frac{1-e^{t/T_1(1-T_1/T_2)}}{e^{t/T_1}-1}. \end{aligned}$$

If $t \ll T_1$, then we can approximate the ratio as $2T_1/T_2 - 1$, as claimed. ■

Thus, an asymmetry in the T_1 and T_2 times does translate to an asymmetry in the occurrence of bit flip and phase flip errors. Note that $p_x = p_y$ indicating that the Y errors are as unlikely as the X errors. We shall refer to the ratio p_z/p_x as the channel asymmetry and denote this parameter by A .

Asymmetric quantum codes use the fact that the phase flip errors are much more likely than the bit flip errors or the combined bit-phase flip errors. Therefore the code has different error correcting capability for handling different type of errors. We require the code to correct many phase flip errors but it is not required to handle the same number of bit flip errors. If we assume a CSS code [4], then we can meaningfully speak of X -distance and Z -distance. A CSS stabilizer code that can detect all X errors up to weight $d_x - 1$ is said to have an X -distance of d_x . Similarly if it can detect all Z errors upto weight $d_z - 1$, then it is said to have a Z -distance of d_z . We shall denote such a code by $[[n, k, d_x/d_z]]_q$ to indicate it is an asymmetric code, see also [19] who was the first to use a notation that allowed to distinguish between X - and Z -distances. We could also view this code as an $[[n, k, \min\{d_x, d_z\}]]_q$ stabilizer code. Further extension of these metrics to an additive non-CSS code is an interesting problem, but we will not go into the details here.

Recall that in the CSS construction a pair of codes are used, one for correcting the bit flip errors and the other for correcting the phase flip errors. Our choice of these codes will be such that the code for correcting the phase flip errors has a larger distance than the code for correcting the bit flip errors. We restate the CSS construction in a form convenient for asymmetric stabilizer codes.

Lemma 2 (CSS Construction [4]): Let C_x, C_z be linear codes over \mathbb{F}_q^n with the parameters $[n, k_x]_q$, and $[n, k_z]_q$ respectively. Let $C_x^\perp \subseteq C_z$. Then there exists an $[[n, k_x + k_z - n, d_x/d_z]]_q$ asymmetric quantum code, where $d_x = \text{wt}(C_x \setminus C_x^\perp)$ and $d_z = \text{wt}(C_z \setminus C_x^\perp)$.

If in the above construction $d_x = \text{wt}(C_x)$ and $d_z = \text{wt}(C_z)$, then we say that the code is pure.

In the theorem above and elsewhere in this paper \mathbb{F}_q denotes a finite field with q elements. We also denote a q -ary narrow-sense primitive BCH code of length $n = q^m - 1$ and design distance δ as $BCH(\delta)$.

III. ASYMMETRIC QUANTUM CODES FROM LDPC CODES

In [10], Ioffe and Mézard used a combination of BCH and LDPC codes to construct asymmetric codes. The intuition being that the stronger LDPC code should be used for correcting the phase flip errors and the BCH code can be used for the infrequent bit flips. This essentially reduces to finding a good LDPC code such that the dual of the LDPC code is contained in the BCH code. They solve this problem by randomly choosing codewords in the BCH code which are of low weight (so that they can be used for the parity check matrix of the LDPC code). However, this method leaves open how good the resulting LDPC code is. For instance, the degree profiles of the resulting code are not regular and there is little control over the final degree profiles of the code. Furthermore, it is not apparent what ensemble or degree profiles one will use to analyze the code.

We propose an alternate scheme that uses LDPC codes to construct asymmetric stabilizer codes. We propose two families of quantum codes based on LDPC codes. In the first case we use LDPC codes for both the X and Z channel while in the second construction we will use a combination of BCH and LDPC codes. But first, we will need the following facts about generalized Reed-Muller codes and finite geometry LDPC codes.

A. Finite Geometry LDPC Codes ([14], [21])

Let us denote by $EG(m, p^s)$ the Euclidean finite geometry over \mathbb{F}_{p^s} consisting of p^{ms} points. For our purposes it suffices to use the fact that this geometry is equivalent to the vector space $\mathbb{F}_{p^s}^m$. A μ -dimensional subspace of $\mathbb{F}_{p^s}^m$ or its coset is called a μ -flat. Assume that $0 \leq \mu_1 < \mu_2 \leq m$. Then we denote by $N_{EG}(\mu_2, \mu_1, s, p)$ the number of μ_1 -flats in a μ_2 -flat and by $A_{EG}(m, \mu_2, \mu_1, s, p)$, the number of μ_2 -flats that contain a given μ_1 -flat. These are given by (see [21])

$$N_{EG}(\mu_2, \mu_1, s, p) = q^{(\mu_2-\mu_1)} \prod_{i=1}^{\mu_1} \frac{q^{\mu_2-i+1} - 1}{q^{\mu_1-i+1} - 1}, \quad (4)$$

$$A_{EG}(m, \mu_2, \mu_1, s, p) = \prod_{i=\mu_1+1}^{\mu_2} \frac{q^{m-i+1} - 1}{q^{\mu_2-i+1} - 1}, \quad (5)$$

where $q = p^s$. Index all the μ_1 -flats from $i = 1$ to $n = N_{EG}(m, \mu_1, s, p)$ as F_i . Let F be a μ_2 -flat in $EG(m, p^s)$. Then

we can associate an incidence vector to F with respect to the μ_1 flats as follows.

$$\mathbf{i}_F = \left\{ i_j \mid \begin{array}{ll} i_j = 1 & \text{if } F_j \text{ is contained in } F \\ i_j = 0 & \text{otherwise.} \end{array} \right\}.$$

Index the μ_2 -flats from $j = 1$ to $J = N_{EG}(m, \mu_2, s, p)$. Construct the $J \times n$ matrix $H_{EG}^{(1)}(m, \mu_2, \mu_1, s, p)$ whose rows are the incidence vectors of all the μ_2 -flats with respect to the μ_1 -flats. This matrix is also referred to as the incidence matrix. Then the type-I Euclidean geometry code from μ_2 -flats and μ_1 -flats is defined to be the null space, i.e., Euclidean dual code) of the \mathbb{F}_p -linear span of $H_{EG}^{(1)}(m, \mu_2, \mu_1, s, p)$. This is denoted as $C_{EG}^{(1)}(m, \mu_2, \mu_1, s, p)$. Let $H_{EG}^{(2)}(m, \mu_2, \mu_1, s, p) = H_{EG}^{(1)}(m, \mu_2, \mu_1, s, p)^t$. Then the type-II Euclidean geometry code $C_{EG}^{(2)}(m, \mu_2, \mu_1, s, p)$ is defined to be the null space of $H_{EG}^{(2)}(m, \mu_2, \mu_1, s, p)$. Let us now consider the μ_2 -flats and μ_1 -flats that do not contain the origin of $EG(m, p^s)$. Now form the incidence matrix of the μ_2 -flats with respect to the μ_1 -flats not containing the origin. The null space of this incidence matrix gives us a quasi-cyclic code in general, which we denote by $C_{EG,c}^{(1)}(m, \mu_2, \mu_1, s, p)$, see [21].

B. Generalized Reed-Muller codes ([12])

Let α be a primitive element in \mathbb{F}_{q^m} . The cyclic generalized Reed-Muller code of length $q^m - 1$ and order ν is defined as the cyclic code with the generator polynomial whose roots α^j satisfy $0 < j \leq m(q-1) - \nu - 1$. The generalized Reed-Muller code is the singly extended code of length q^m . It is denoted as $GRM_q(\nu, m)$. The dual of a GRM code is also a GRM code [2], [3], [12]. It is known that

$$GRM_q(\nu, m)^\perp = GRM_q(\nu^\perp, m), \quad (6)$$

where $\nu^\perp = m(q-1) - 1 - \nu$.

Let C be a linear code over \mathbb{F}_{q^s} . Then we define $C|_{\mathbb{F}_q}$, the subfield subcode of C over \mathbb{F}_q as the codewords of C which are entirely in \mathbb{F}_q^n , (see [9, pages 116-120]). Formally this can be expressed as

$$C|_{\mathbb{F}_q} = \{c \in C \mid c \in \mathbb{F}_q^n\}. \quad (7)$$

Let $C \subseteq \mathbb{F}_{q^s}^n$. The trace code of C over \mathbb{F}_q is defined as

$$\text{tr}_{q^s/q}(C) = \{\text{tr}_{q^s/q}(c) \mid c \in C\}. \quad (8)$$

There are interesting relations between the trace code and the subfield subcode. One of which is the following result which we will need later.

Lemma 3: Let $C \subseteq \mathbb{F}_{q^s}^n$. Then $C|_{\mathbb{F}_q}$, the subfield subcode of C is contained in $\text{tr}_{q^s/q}(C)$, the trace code of C . In other words

$$C|_{\mathbb{F}_q} \subseteq \text{tr}_{q^s/q}(C).$$

Proof: Let $c \in C|_{\mathbb{F}_q} \subseteq \mathbb{F}_q^n$ and $\alpha \in \mathbb{F}_{q^s}$. Then $\text{tr}_{q^s/q}(\alpha c) = c \text{tr}_{q^s/q}(\alpha)$ as $c \in \mathbb{F}_q^n$. Since trace is a surjective form, there exists some $\alpha \in \mathbb{F}_{q^s}$, such that $\text{tr}_{q^s/q}(\alpha) = 1$. This implies that $c \in \text{tr}_{q^s/q}(C)$. Since c is an arbitrary element in $C|_{\mathbb{F}_q}$ it follows that $C|_{\mathbb{F}_q} \subseteq \text{tr}_{q^s/q}(C)$. ■

Let $q = p^s$, then the Euclidean geometry code of order r over $EG(m, p^s)$ is defined as the dual of the subfield subcode of $GRM_q((q-1)(m-r-1), m)$, [3, page 448]. The type-I LDPC code $C_{EG}^{(1)}(m, \mu, 0, s, p)$ code is an Euclidean geometry code of order $\mu-1$ over $EG(m, p^s)$, see [21]. Hence its dual is the subfield subcode of $GRM_q((q-1)(m-\mu), m)$ code. In other words,

$$C_{EG}^{(1)}(m, \mu, 0, s, p)^\perp = GRM_q((q-1)(m-\mu), m)|_{\mathbb{F}_p}. \quad (9)$$

Further, Delsarte's theorem [6] tells us that

$$\begin{aligned} C_{EG}^{(1)}(m, \mu, 0, s, p) &= GRM_q((q-1)(m-\mu), m)|_{\mathbb{F}_p}^\perp, \\ &= \text{tr}_{q/p}(GRM_q((q-1)(m-\mu), m)^\perp) \\ &= \text{tr}_{q/p}(GRM_q(\mu(q-1)-1, m)). \end{aligned}$$

Hence, $C_{EG}^{(1)}(m, \mu, 0, s, p)$ code can also be related to $GRM_q(\mu(q-1)-1, m)$ as

$$C_{EG}^{(1)}(m, \mu, 0, s, p) = \text{tr}_{q/p}(GRM_q(\mu(q-1)-1, m)). \quad (10)$$

C. New families of asymmetric quantum codes

With the previous preparation we are now ready to construct asymmetric quantum codes from finite geometry LDPC codes.

Theorem 4 (Asymmetric EG LDPC Codes): Let p be a prime, with $q = p^s$ and $s \geq 1, m \geq 2$. Let $1 < \mu_z < m$ and $m - \mu_z + 1 \leq \mu_x < m$. Then there exists an

$$[[p^{ms}, k_x + k_z - p^{ms}, d_x/d_z]]_p$$

asymmetric EG LDPC code, where

$$k_x = \dim C_{EG}^{(1)}(m, \mu_x, 0, s, p); \quad k_z = \dim C_{EG}^{(1)}(m, \mu_z, 0, s, p).$$

For the distances $d_x \geq A_{EG}(m, \mu_x, \mu_x-1, s, p) + 1$ and $d_z \geq A_{EG}(m, \mu_z, \mu_z-1, s, p) + 1$ hold.

Proof: Let $C_z = C_{EG}^{(1)}(m, \mu_z, 0, s, p)$. Then from equation (10) we have

$$C_z = \text{tr}_{q/p}(GRM_q(\mu_z(q-1)-1, m)).$$

By Lemma 3 we know that

$$\begin{aligned} C_z &\supseteq GRM_q(\mu_z(q-1)-1, m)|_{\mathbb{F}_p}, \\ C_z &\supseteq GRM_q((q-1)(m-(m-\mu_z+1)), m)|_{\mathbb{F}_p}, \end{aligned}$$

where the last inclusion follows from the nesting property of the generalized Reed-Muller codes. For any order μ_x such that $m - \mu_z + 1 \leq \mu_x < m$, let $C_x = C_{EG}^{(1)}(m, \mu_x, 0, s, p)$. Then C_x is an LDPC code whose dual $C_x^\perp = GRM_q((q-1)(m-\mu_x), m)|_{\mathbb{F}_p}$ is contained in C_z . Thus we can use Lemma 2 to form an asymmetric code with the parameters

$$[[p^{ms}, k_x + k_z - p^{ms}, d_x/d_z]]_p$$

The distance of C_z and C_x are at lower bounded as $d_x \geq A_{EG}(m, \mu_x, \mu_x-1, s, p) + 1$ and $d_z \geq A_{EG}(m, \mu_z, \mu_z-1, s, p) + 1$ (see [21]). ■

In the construction just proposed, we should choose C_z to be a stronger code compared to C_x . We have given the construction over a nonbinary alphabet even though the case $p = 2$ might be of particular interest.

Our next construction makes use of the cyclic finite geometry codes. Our goal will be to find a small BCH code whose dual is contained in a cyclic Euclidean geometry LDPC code. For solving this problem we need to know the cyclic structure of $C_{EG,c}^{(1)}(m, \mu, 0, s, p)$. Let α be a primitive element in $\mathbb{F}_{p^{ms}}$. Then the roots of generator polynomial of $C_{EG,c}^{(1)}(m, \mu, 0, s, p)$ are given by [11, Theorem 6], see also [13], [15]. Now,

$$Z = \{\alpha^h \mid 0 < \max_{0 \leq l < s} W_{p^s}(hp^l) \leq (p^s - 1)(m - \mu)\},$$

where $W_q(h)$ is the q -ary weight of $h = h_0 + h_1q + \dots + h_kq^{k-1}$, i.e., $W_q(h) = \sum h_i$. The finite geometry code $C_{EG,c}^{(1)}(m, \mu, 0, s, p)$ is actually an $(\mu - 1, p^s)$ Euclidean geometry code. The roots of the generator polynomial of the dual code are given by

$$Z^\perp = \{\alpha^h \mid \min_{0 \leq l < s} W_{p^s}(hp^l) < \mu(p^s - 1)\}.$$

In fact, the dual code is the even-like subcode of a primitive polynomial code of length $p^{ms} - 1$ over \mathbb{F}_p and order $m - \mu$, whose generator polynomial, by [13, Theorem 6], has the roots

$$Z_p = \{\alpha^h \mid 0 < \min_{0 \leq l < s} W_{p^s}(hp^l) < \mu(p^s - 1)\}.$$

Thus $Z^\perp = Z_p \cup \{0\}$. Now by [13, Theorem 11], Z_p and therefore Z^\perp contain the sequence of consecutive roots, $\alpha, \alpha^2, \dots, \alpha^{\delta_0 - 1}$, where $\delta_0 = (R + 1)p^{Qs} - 1$ and $m(p^s - 1) - (m - \mu)(p^s - 1) = Q(p^s - 1) + R$. Simplifying, we see that $R = 0$ and $Q = \mu$ giving $\delta_0 = p^{\mu s} - 1$. It follows that

$$\begin{aligned} C_{EG,c}^{(1)}(m, \mu, 0, s, p)^\perp &= \text{GRM}_q(m, (q - 1)(m - \mu))|_{\mathbb{F}_p} \\ &\subseteq \text{BCH}(\delta_0). \end{aligned}$$

Thus we have solved the problem of construction of the asymmetric stabilizer codes in a dual fashion to that of [10]. Instead of finding an LDPC code whose parity check matrix is contained in a given BCH code, we have found a BCH code whose parity check matrix is contained in a given finite geometry LDPC code. This gives us the following result.

Theorem 5 (Asymmetric BCH-LDPC stabilizer codes):

Let $C_z = C_{EG,c}^{(1)}(m, \mu, 0, s, p)$ and $\delta \leq \delta_0 = p^{\mu s} - 1$. Let $n = p^{ms} - 1$ and $C_x = \text{BCH}(\delta) \subseteq \mathbb{F}_p^n$. Then there exists an

$$[[n, k_x + k_z - n, d_x/d_z]]_p$$

asymmetric stabilizer code where $d_z \geq A_{EG}(m, \mu, \mu - 1, s, p)$, $d_x \geq \delta$ and $k_x = \dim C_x$, $k_z = \dim C_z$.

Perhaps an example will be helpful at this juncture.

Example 6: Let $m = s = p = 2$ and $\mu = 1$. Then $C_{EG,c}^{(1)}(2, 1, 0, 2, 2)$ is a cyclic code whose generator polynomial has roots given by

$$\begin{aligned} Z &= \{\alpha^h \mid 0 < \max_{0 \leq l < 2} W_{2^2}(2^l h) \leq (m - \mu)(p^s - 1) = 3\} \\ &= \{\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^6, \alpha^8, \alpha^9, \alpha^{12}\}. \end{aligned}$$

As there are 4 consecutive roots and $|Z| = 8$, it defines a $[[15, 7, \geq 5]]$ code. The roots of the generator polynomial of

the dual code are given by

$$\begin{aligned} Z^\perp &= \{\alpha^h \mid 0 < \min_{0 \leq l < 2} W_{2^2}(2^l h) \leq \mu(p^s - 1) = (2^2 - 1)\} \\ &= \{\alpha^0, \alpha^1, \alpha^2, \alpha^4, \alpha^5, \alpha^8, \alpha^{10}\}. \end{aligned}$$

We see that Z^\perp has two consecutive roots excluding 1, therefore the dual code is contained in a narrow-sense BCH code with design distance 3. Note that $p^{\mu s} - 1 = 3$. Thus we can choose $C_x = \text{BCH}(3)$ and $C_z = C_{EG,c}^{(1)}(2, 1, 0, 2, 2)$ and apply Lemma 2 to construct a $[[15, 3, 3/5]]_2$ asymmetric code.

We can also state the above construction as in [10], that is given a primitive BCH code of design distance δ , find an LDPC code whose dual is contained in it. It must be pointed out that in case of asymmetric codes derived from LDPC codes, the asymmetry factor d_x/d_z is not as indicative of the code performance as in the case of bounded distance decoders. For $m = p = 2$, we can derive explicit relations for the parameters of the codes.

Corollary 7: Let $C = C_{EG,c}^{(1)}(2, 1, 0, s, 2)$ and $\delta = 2t + 1 \leq 2^s - 1$. Then there exists an

$$[[2^{2s} - 1, 2^{2s} - 3^s - s(\delta - 1), \delta/2^s + 1]]_2$$

asymmetric stabilizer code.

Proof: The parameters of C are $[2^{2s} - 1, 2^{2s} - 3^s, 2^s + 1]_2$, see [15]. Since C^\perp is contained in a BCH code of length $2^{2s} - 1$ whose design distance $\delta \leq 2^s - 1$, we can compute the dimension of the BCH code as $2^{2s} - 1 - s(\delta - 1)$, see [17, Corollary 8]. By Lemma 2 the quantum code has the dimension $2^{2s} - 3^s - s(\delta - 1)$. ■

Example 8: For $m = p = 2$ and $s = 4$ we can obtain a $[255, 175, 17]$ LDPC code. We can choose any BCH code with design distance $\delta \leq 2^4 - 1 = 15$ to construct an asymmetric code. Table I lists possible codes.

TABLE I
ASYMMETRIC BCH-LDPC STABILIZER CODES

s	δ	Code [[n, k, d _x /d _z]] ₂	Asymmetry d _z /d _x	Rate
4	15	[[255, 119, 15/17]] ₂	≈ 1	0.467
4	13	[[255, 127, 13/17]] ₂	≈ 1.25	0.498
4	11	[[255, 135, 11/17]] ₂	≈ 1.5	0.529
4	9	[[255, 143, 9/17]] ₂	≈ 2	0.561
4	7	[[255, 151, 7/17]] ₂	≈ 2.5	0.592
4	5	[[255, 159, 5/17]] ₂	≈ 3	0.624
4	3	[[255, 167, 3/17]] ₂	≈ 6	0.655

IV. PERFORMANCE RESULTS

We now study the performance of the codes constructed in the previous section. Due to space constraints the discussion will be rather brief, but more details will be supplied in a forthcoming paper. We assume that the overall probability of error in the channel is given by p , while the individual probabilities of X , Y , and Z errors are $p_x = p/(A + 2)$, $p_y = p/(A + 2)$ and $p_z = pA/(A + 2)$ respectively. The exact performance would require us to simulate a 4-ary channel and also account for the fact that some errors can be estimated modulo the stabilizer. However, we do not account for this

and in that sense these results provide an upper bound on the actual error rates. The 4-ary channel can be modeled as two binary symmetric channels – one modeling the bit flip channel and the other the phase flip channel. For exact performance, these two channels should be dependent, however, a good approximation is to model the channel as two independent BSCs with cross over probabilities $p_x + p_y = 2p/(A + 2)$ and $p_y + p_z = p(A + 1)/(A + 2)$. In this case the overall error rate in the quantum channel is the sum of the error rates in the two BSCs. While this approach is going to slightly overestimate the error rates, nonetheless it is useful and has been used before [16]. Since the X-channel uses a BCH code and decoded using a bounded distance decoder, we can just compute P_e^x the X error rate, in closed form. The error rate in the Z channel, P_e^z is obtained through simulations. The overall error rate is

$$P_e = 1 - (1 - P_e^x)(1 - P_e^z) = P_e^x + P_e^z - P_e^x P_e^z \approx P_e^x + P_e^z.$$

The LDPC code was decoded using the hard decision bit flipping algorithm given in [14]. The maximum number of iterations for decoding is set to 50. In Figure 1 we see the performance of $[[255, 159, 5/17]]$ as the channel asymmetry is varied from 1 to 100. We can clearly see the improvement as the channel asymmetry increases.

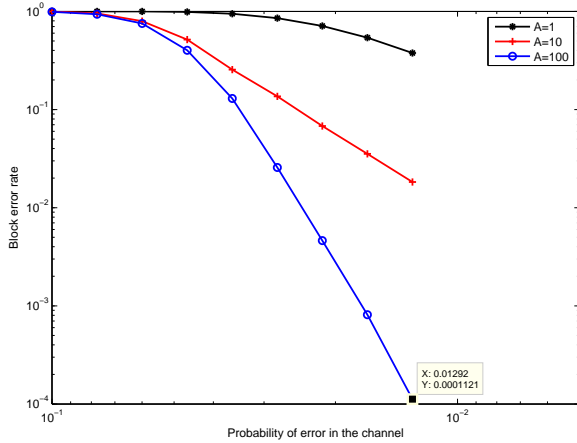


Fig. 1. Performance of $[[255, 159, 5/17]]$ code for $A = 1, 10, 100$

The question naturally raises how do these codes compare with the codes proposed in [10]. Strictly speaking both constructions have regimes where they can perform better than the other. But it appears that the algebraically constructed asymmetric codes have the following benefits with respect to the randomly constructed ones of [10].

- They give comparable performance and higher data rates with shorter lengths.
- The benefits of classical algebraic LDPC codes are inherited, giving for instance lower error floors compared to the random constructions.
- The code construction is systematic.

Our codes also offer flexibility in the rate and performance of the code because we can choose many possible BCH codes for a given finite geometry LDPC code or vice versa. The flip side however is that the codes given here have higher complexity of decoding.

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